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1987 J. Phys. A: Math. Gen. 20 1951

(http://iopscience.iop.org/0305-4470/20/8/015)

J. Phys. A: Math. Gen. **20** (1987) 1951-1959. Printed in the UK

Symmetries and constants of motion for new AKNS hierarchies

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Received 16 July 1986

Abstract. Two sets of infinite number of symmetries, their Lie algebra properties and constants of motion for a class of non-linear evolution equations associated with non-isospectral deformations of the AKNS spectral problem are constructed.

1. Introduction

In Li and Zhu (1986), they investigated the symmetries for a class of integrable non-linear evolution equations (NEE)

$$u_t = K_l(u) \qquad K_l = \phi^l K_0 \qquad l = 0, 1, 2, \ldots \tag{1.1}$$

where $u = \begin{pmatrix} q \\ r \end{pmatrix}$, $K_0 = \begin{pmatrix} -\mathrm{i}q \\ \mathrm{i}r \end{pmatrix}$ and ϕ is an operator

$$\phi = \frac{1}{\mathrm{i}}\begin{pmatrix} -D+2qD^{-1}r & 2qD^{-1}q \\ -2rD^{-1}r & D-2rD^{-1}q \end{pmatrix} \tag{1.2}$$

$$D = \partial/\partial x \qquad D^{-1}D = DD^{-1} = I.$$

They constructed two sets of infinite number of symmetries in terms of the operator ϕ in (1.2)

$$K_n = \phi^n K_0 \tag{1.3}$$

$$\tau_n^l = \phi^n(ltK_{l-1} + \sigma_0) = ltK_{n+l-1} + \sigma_n \qquad \sigma_0 = xK_0 \qquad \sigma_n = \phi^n \sigma_0 \tag{1.4}$$

and showed that a graded infinite-dimensional Lie algebra exists for these symmetries

$$[K_n K_m] = 0 \tag{1.5}$$

$$[K_n \tau_m^l] = nK_{n+m-1} \tag{1.6}$$

$$[\tau_n^l \tau_m^l] = (n-m)\tau_{n+m-1}^l. \tag{1.7}$$

The symmetries of (1.1) is an algebraic expression $\tau(u)$ that satisfies the following linearised equations:

$$\tau_t = K_l'[\tau] \tag{1.8}$$

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where $K_l'[\tau]=(\partial/\partial\varepsilon)K_l(u+\varepsilon\tau)|_{\varepsilon=0}$ is the Gateaux derivative of K_l in the direction of τ and the Lie bracket between any two symmetries is defined as follows:

$$[A, B]=A'[B]-B'[A]. \tag{1.9}$$

The equations (1.1) are well known to be the integrable NEE associated with the isospectral deformations of the AKNS spectral problem (Ablowitz *et al* 1974)

$$\varphi_x=M\varphi \qquad M=\begin{pmatrix}-\mathrm{i}\xi/2 & q\\ r & \mathrm{i}\xi/2\end{pmatrix} \qquad \varphi=\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}. \tag{1.10}$$

Like most other integrable NEE, (1.1) can be expressed as the Hamiltonian systems and their infinite number of symmetries K_n (called classical ones) yield the constants of motion which are in involution through Noether's theorem (Magri 1980), namely there exist potentials I_n such that

$$\theta \operatorname{grad} I_n=K_n \qquad \theta=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \tag{1.11}$$

$$I_{nt}=0 \qquad \{I_n I_m\}\equiv\langle \operatorname{grad} I_n \theta \operatorname{grad} I_m\rangle=\langle K_n\theta K_m\rangle=0. \tag{1.12}$$

But for symmetries of τ_n^l in (1.4) (called new ones), there no longer exist corresponding constants of motion. One reason is that the symmetry transformations associated with τ_n^l change the actions of NEE (1.1) and are non-Noether's (Cheng and Li 1986).

The other interesting fact is that for the hierarchy of NEE $u_t=\tau$ (generated by symmetry τ), when $\tau=K_n$ the equations $u_t=K_n$ actually belong to the class of NEE (1.1). However, when $\tau=\tau_n^l$ the equations

$$u_t=\tau_n^l \tag{1.13}$$

are nothing but the NEE corresponding to the non-isospectral deformations of the AKNS spectral problem (1.10), namely they are equivalent to the Lax formulations with the same spectral problem (1.10) in the usual isospectral case, such that its spectrum ξ be no longer invariant but $\xi_t=\xi^n$. This fact has been firstly pointed out by Li and Zhu (1986). The NEE (1.13) are not usual Hamiltonian (except $n=0$), but they can be solved by the inverse scattering transform (Calogero and Degasperis 1978, Li 1981). From a mathematical point of view, they enlarge classes of solvable NEE. Physically, they can be regarded as the dynamical systems with soliton equations (corresponding to the dispersion relations $\Omega(\xi)=lt\xi^{n+l-1}$) plus extra terms ($\sigma_n$). Our interest in this paper is to discuss the symmetry structure and constants of motion for such NEE (1.13). We shall see that they have different features to the Hamiltonian systems (1.1). Before entering into the main discussions, we shall summarise the notation and some notions.

2. Basic notation and notions

Let S be a space of the vector $u=\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ such that v_1, v_2 are sufficiently smooth functions of x and t. They and any possible derivative vanish as fast as one wants for $|x|\to\infty$. S^* is the dual space of S. For $\alpha\in S^*$ and $v\in S$ $\langle\alpha v\rangle$ denotes the application of the linear functional given by α to v. ϕ in (1.2) is the operator from S to S and with respect to this duality is denoted by ϕ^*: $S^*\to S^*$. It satisfies $\phi\theta=\theta\phi^*$, where θ in (1.11) is an operator from S^* to S and is obviously impletic (Magri 1980, Fuchssteiner and Fokas 1981). The further properties for ϕ are that ϕ is a hereditary symmetry and a strong symmetry for NEE (1.1) (see Li and Zhu 1986), i.e.

$$\phi'[\phi u]v - \phi'[\phi v]u = \phi(\phi'[u]v - \phi'[v]u) \tag{2.1}$$

$$\phi'[K_l] = [K_l' \phi] \qquad l = 0, 1, 2, \ldots \tag{2.2}$$

for any $u, v \in S$, the bracket for two operators means $[FG] = FG - GF$. From equations (2.1) and (2.2), we have the following lemma.

Lemma 1.

$$(\phi^k)'[K_l] = [K_l' \phi^k] \qquad k, l = 0, 1, 2 \ldots \tag{2.3}$$

$$(\phi^k)'[\tau_n^l] = [\tau_n^{l\prime} \phi^k] + k\phi^{n+k-1} \qquad k, n = 0, 1, 2 \ldots. \tag{2.4}$$

Proof (2.3).

$$\begin{aligned}(\phi^k)'[K_l] &= \sum_{j=1}^{k} \phi^{k-j}\phi'[K_l]\phi^{j-1} \\ &= \sum_{j=1}^{k} \phi^{k-j}K_l'\phi^j - \sum_{j=1}^{k} \phi^{k-j+1}K_l'\phi^{j-1} \\ &= [K_l' \phi^k].\end{aligned}$$

Proof (2.4).
For $k = 1$, $n = 0$. It has been proved by Li and Zhu (1986) that

$$\phi'[\tau_0^l] = [\tau_0^l \phi] + I$$

where I is the unit operator. So, by means of (2.1), we have the following.

For $k = 1$, $n > 0$. By acting on any $v \in S$

$$\begin{aligned}(\phi'[\tau_n^l] - [\tau_n^{l\prime}\phi])v &= \phi'[\tau_n^l]v - \tau_n^{l\prime}[\phi v] + \phi(\tau_n^{l\prime}[v]) \\ &= \phi'[\phi\tau_{n-1}]v - \phi'[\phi v]\tau_{n-1}^l - \phi\tau_{n-1}^{l\prime}[\phi v] + \phi(\phi'[v]\tau_{n-1}^l) + \phi^2\tau_{n-1}^{l\prime}[v] \\ &= \phi((\phi'[\tau_{n-1}^l] - [\tau_{n-1}^{l\prime}\phi])v).\end{aligned}$$

Hence $\phi'[\tau_n^l] = [\tau_n^{l\prime} \phi] + \phi^n$.

For $k > 1$, $n > 0$

$$\begin{aligned}(\phi^k)'[\tau_n^l] &= \sum_{j=1}^{k} \phi^{k-j}\phi'[\tau_n^l]\phi^{j-1} \\ &= [\tau_n^{l\prime} \phi^k] + k\phi^{n+k-1}.\end{aligned}$$

Thus, we complete our proofs.

Here and in the following, for simplicity, we assume $\tau = \tau_{N+1}^l$ and consider the following NEE

$$u_t = \tau \qquad \tau = \tau_{N+1}^l \qquad N = -1, 0, 1, 2, \ldots \tag{2.5}$$

where u_t indicates the total differentiation with respect to t. So, for any functional $F(u, t)$, $F_t = \partial F/\partial t + F'[\tau]$ and the definitions of the symmetry $\bar{\tau}$ and strong symmetry $\bar{\phi}$ for (2.5) are formally the same as usual (e.g. see Li and Zhu 1986), namely

$$\bar{\tau}_t = \tau'[\tau] \tag{2.6}$$

$$\bar{\phi}_t = [\tau' \bar{\phi}]. \tag{2.7}$$

If $\bar{\tau}$ and $\bar{\phi}$ are explicitly dependent on t, then (2.6) and (2.7) are equivalent to

$$\partial\bar{\tau}/\partial t = [\tau\bar{\tau}] \tag{2.8}$$

$$\frac{\partial\bar{\phi}}{\partial t} + \bar{\phi}'[\bar{\tau}] = [\tau' \bar{\phi}]. \tag{2.9}$$

This notation means that the strong symmetry ϕ generates the symmetries of (2.5), i.e. if $\bar{\tau}$ is a symmetry of (2.5), then $\phi\bar{\tau}$ is also a symmetry.

We also have the following corollary.

Corollary.

$$K_{mt} = \tau'[K_m] + mK_{m+N} \tag{2.10}$$

$$\tau^l_{nt} = \tau'[\tau^l_n] + lK_{n+l-1} + (n - N - 1)\tau^l_{n+N} \tag{2.11}$$

$$(\phi^k)_t = [\tau' \phi^k] + k\phi^{k+N}. \tag{2.12}$$

The corollary can be proved from the Lie brackets (1.5), (1.6) and lemma 1.

3. Symmetries for NEE (2.5)

In this section, we shall construct a strong symmetry for (2.5) and then by means of strong symmetry, we can obtain two sets of infinite number of symmetries. Finally we give the Lie brackets among these symmetries.

Theorem 1. Let

$$f(\xi, t) = \xi(1 + Nt\xi^N)^{-1/N}. \tag{3.1}$$

Then $\bar{\phi} = f(\phi, t)$ is a strong symmetry for NEE (2.5), i.e. it satisfies (2.7)

Proof. One notices that the function $f(\xi, t)$ satisfies

$$f(\xi, t) = \begin{cases} \xi - t & N = -1 \\ e^{-t}\xi & N = 0 \\ \sum_{k=0}^{\infty} f_{1+Nk}\xi^{1+Nk} & N \geq 1 \end{cases} \tag{3.2}$$

and

$$(\partial/\partial t)f(\xi, t) = -f^{N+1} \tag{3.3}$$

where the coefficients f_{1+Nk} satisfy

$$\begin{aligned} &f_1 = 1 \\ &(f_{1+Nk})_t + (1 + N(k-1))f_{1+N(k-1)} = 0 \qquad k = 1, 2, 3, \ldots. \end{aligned} \tag{3.4}$$

According to these properties of $f(\xi, t)$, one can check that our operator satisfies (2.7).

Theorem 2. Let $\bar{K}_0 = K_0$, $\bar{\tau}_0 = \tau - K_l$ and

$$\bar{K}_n = \bar{\phi}^n \bar{K}_0 \qquad n = 1, 2, 3, \dots \tag{3.5}$$

$$\bar{\tau}_n = \bar{\phi}^n \bar{\tau}_0 \qquad n = 1, 2, 3, \dots \tag{3.6}$$

then $\bar{K}_n$ and $\bar{\tau}_n$ are symmetries for NEE (2.5).

Proof. As we know, the strong symmetry $\bar{\phi}$ for (2.5) generates its symmetries, so what we need to do is just to prove that $\bar{K}_0$ and $\bar{\tau}_0$ are symmetries. It is obvious, from (2.10), that $\bar{K}_0$ is a symmetry, while for $\bar{\tau}_0$, one can check from (2.10) and (2.11) that

$$\begin{aligned}\bar{\tau}_{0t} &= \tau_t - K_{lt} = \tau'[\tau] + lK_{l+N} - \tau'[K_l] - lK_{l+N} \\ &= \tau'[\bar{\tau}_0].\end{aligned}$$

Theorem 3. For two sets of symmetries (3.5) and (3.6), there exists a graded infinite-dimensional Lie algebra

$$[\bar{K}_m \bar{K}_n] = 0 \tag{3.7}$$

$$[\bar{K}_m \bar{\tau}_n] = m\bar{K}_{m+n+N} \tag{3.8}$$

$$[\bar{\tau}_m \bar{\tau}_n] = (m-n)\bar{\tau}_{m+n+N}. \tag{3.9}$$

Before we prove this theorem, we need to prove the following lemma.

Lemma 2.

$$\bar{\phi}'[K_n] = [\bar{K}'_n \bar{\phi}] \tag{3.10}$$

$$\bar{\phi}'[\bar{\tau}_n] = [\bar{\tau}'_n \bar{\phi}] + \bar{\phi}^{n+N+1}. \tag{3.11}$$

Proof. For $N \geqslant 1$, from (3.2) we have

$$\bar{\phi} = \sum_{k=0}^{\infty} f_{1+Nk} \phi^{1+Nk}. \tag{3.12}$$

Similarly, $\bar{\phi}^n$ can also be expressed as the series of ϕ in the form

$$\bar{\phi}^n = \sum a_j \phi^j.$$

So $\bar{K}_n$ and $\bar{\tau}_n$ have the following forms:

$$\bar{K}_n = \sum a_j K_j$$

$$\bar{\tau}_n = \sum a_j (\tau^l_{N+1+j} - K_{l+j}).$$

According to lemma 1, we have

$$\begin{aligned}\bar{\phi}'[\bar{K}_n] &= \sum_k \sum_j f_{1+Nk} a_j (\phi^{1+Nk})'[K_j] \\ &= \sum_k \sum_j f_{1+Nk} a_j [K'_j \phi^{1+Nk}] \\ &= [\bar{K}'_n \bar{\phi}]\end{aligned}$$

and

$$\bar{\phi}'[\bar{\tau}_n]=\sum_k\sum_j f_{1+Nk}a_j(\phi^{1+Nk})'[\tau^l_{N+1+j}-K_{l+j}]$$

$$=\sum_k\sum_j f_{1+Nk}a_j\{[\tau'^{l}_{N+1+j}-K'_{l+j}\phi^{1+Nk}]+(1+Nk)\phi^{1+N(k+1)+j}\}$$

$$=[\bar{\tau}'_n\bar{\phi}]+\bar{\phi}^n\sum_k(1+Nk)f_{1+Nk}\phi^{1+N(k+1)}$$

$$=[\bar{\tau}'_n\bar{\phi}]-\bar{\phi}^n\frac{\partial}{\partial t}f(\phi,t)$$

$$=[\bar{\tau}'_n\bar{\phi}]+\bar{\phi}^{n+N+1}$$

The remains (in the cases of $N=-1$ and $N=0$) can be proved similarly.

We now prove theorem 3.

Proof (3.7). Notice that $\bar{K}_n$ are combinations of K_n, so the commutability among K_n implies the commutability among $\bar{K}_n$.

Proof (3.8). In accordance with lemma 2, we have the following.

For $m=0$, $n=0$

$$[\bar{K}_0\bar{\tau}_0]=[K_0\tau^l_{N+1}-K_l]=0.$$

For $m>0$, $n=0$

$$\begin{aligned}[\bar{K}_m\bar{\tau}_0]&=\bar{K}'_m[\bar{\tau}_0]-\bar{\tau}'_0[\bar{K}_m]\\&=\bar{\phi}'[\bar{\tau}_0]\bar{K}_{m-1}-\bar{\tau}'_0[\bar{\phi}\bar{K}_{m-1}]+\bar{\phi}\bar{K}'_{m-1}[\bar{\tau}_0]\\&=-\bar{\phi}\bar{\tau}'_0[\bar{K}_{m-1}]+\bar{\phi}^{N+1}\bar{K}_{m-1}+\bar{\phi}\bar{K}'_{m-1}[\bar{\tau}_0]\\&=\bar{\phi}([\bar{K}_{m-1}\bar{\tau}_0])+\bar{K}_{m+N}\\&=m\bar{K}_{m+N}.\end{aligned}$$

For $m>0$, $n>0$

$$\begin{aligned}[\bar{K}_m\bar{\tau}_n]&=\bar{K}'_m[\bar{\tau}_n]-\bar{\tau}'_n[\bar{K}_m]\\&=\bar{K}'_m[\bar{\phi}\bar{\tau}_{n-1}]-\bar{\phi}\bar{\tau}'_{n-1}[\bar{K}_m]-\bar{\phi}'[\bar{K}_m]\bar{\tau}_{n-1}\\&=\bar{\phi}([\bar{K}_m\bar{\tau}_{n-1}])\\&=\bar{\phi}^n([\bar{K}_m\bar{\tau}_0])\\&=m\bar{K}_{m+n+N}.\end{aligned}$$

Proof (3.9). Similarly, we have the following.

For $m=0$, $n=0$

$$[\bar{\tau}_0\bar{\tau}_0]=0.$$

For $m>0$, $n=0$

$$\begin{aligned}[\bar{\tau}_m\bar{\tau}_0]&=\bar{\phi}\bar{\tau}'_{m-1}[\bar{\tau}_0]+\bar{\phi}'[\bar{\tau}_0]\bar{\tau}_{m-1}-\bar{\tau}'_0[\bar{\phi}\bar{\tau}_{m-1}]\\&=\bar{\phi}\bar{\tau}'_{m-1}[\bar{\tau}_0]-\bar{\phi}\bar{\tau}'_0[\bar{\tau}_{m-1}]+\bar{\phi}^{N+1}\bar{\tau}_{m-1}\\&=\bar{\phi}([\bar{\tau}_{m-1}\bar{\tau}_0])+\bar{\tau}_{m+N}\\&=m\bar{\tau}_{m+N}.\end{aligned}$$

For $m>0$, $n>0$

$$\begin{aligned}[\bar\tau_m\,\bar\tau_n] &= \bar\tau'_m[\bar\phi\bar\tau_{n-1}] - \bar\phi'[\bar\tau_m]\bar\tau_{n-1} - \bar\phi\bar\tau'_{n-1}[\bar\tau_m] \\ &= \bar\phi\bar\tau'_m[\bar\tau_{n-1}] - \bar\phi\bar\tau'_{n-1}[\bar\tau_m] - \bar\phi^{m+N+1}\bar\tau_{n-1} \\ &= \bar\phi([\bar\tau_m\bar\tau_{n-1}]) - \bar\tau_{m+n+N} \\ &= \bar\phi^n([\bar\tau_m\bar\tau_0]) - n\bar\tau_{m+n+N} \\ &= (m-n)\bar\tau_{m+n+N}.\end{aligned}$$

4. Constants of motion

For NEE (2.5), the constant of motion $\bar I$ means that u satisfies (2.5) and implies $\bar I_t=0$.

Theorem 4. Define

$$\bar I_n = \begin{cases} \sum_{k=0}^{n} C_n^k(-t)^{n-k} I_k & N=-1 \\ \exp[-(n+1)t] I_n & N=0 \\ \sum_{l=0}^{\infty} b_{n+Nl} I_{n+Nl} & N\geqslant 1 \end{cases} \tag{4.1}$$

for $n=0,1,2,\ldots$, where I_n are constants of motion for NEE (1.1) which were defined in (1.11). $C_n^k = n!/k!(n-k)!$ and b_{n+Nl} are coefficients of

$$g_n(\xi,t) = f^n\frac{\partial f}{\partial\xi} = \sum_{l=0}^{\infty} b_{n+Nl}\xi^{n+Nl}. \tag{4.2}$$

Then, $\bar I_n$ are constants of motion for NEE (2.5) which are in involution.

Proof. From the lemma in Cheng and Li (1986), we know that when $u_t=\tau$, I_n in (1.11) satisfy

$$I_{nt} = I'_n[\tau] = (n+N+1)I_{n+N}. \tag{4.3}$$

So, for $N=-1$ and $N=0$, it can be checked that $\bar I_n$ defined in (4.1) are constants of motion.

For $N\geqslant 1$, one notices that the coefficients b_{n+Nl} in (4.2) satisfy

$$\begin{aligned} &b_n = 1 \\ &(b_{n+Nl})_t + (n+Nl+1)b_{n+N(l-1)} = 0. \end{aligned} \tag{4.4}$$

This implies that $\bar I_{nt}=0$ if $u_t=\tau$.

On the other hand, $\bar I_n$ are combinations of I_n in each case. Thus the commutability of I_n under the meaning of the Poisson brackets (1.12) implies that $\bar I_n$ are in involution with the same meaning

$$\{\bar I_m\,\bar I_n\} = \langle \operatorname{grad}\bar I_m, \theta\operatorname{grad}\bar I_n\rangle = 0. \tag{4.5}$$

So, we complete our proof of theorem 4.

In accordance with the inverse scattering transform theories of (1.10), we finally give the expressions for the constants of motion $\bar{I}_n$ in terms of scattering data. We assume here $r=-q^*$ to be an example, the asterisk meaning the complex conjugate. From the paper of Newell (1980), I_n have the following expressions

$$I_n=\sum_{k=0}^{M}\frac{\xi_k^{n+1}-\xi_k^{*a+1}}{n+1}+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}\xi^n\ln|a|^2\,\mathrm{d}\xi \tag{4.6}$$

where ξ_k and ξ_k^* are discrete eigenvalues and $a(\xi)$ is the transmission coefficient of the spectral problem (1.10). By means of the definition of the function f and theorem 4, we have

$$\bar{I}_n=\sum_{k=0}^{M}\frac{f^{n+1}(\xi_k,t)-f^{n+1}(\xi_k^*,t)}{n+1}+\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{+\infty}f^n\ln|a|^2\,\mathrm{d}f. \tag{4.7}$$

5. Concluding remarks

In this paper, we have investigated the symmetries, their Lie algebra properties and constants of motion for NEE (2.5) which are associated with non-isospectral deformations of the AKNS spectral problem (1.10) ($\xi_t=\xi^{N+1}$) and which can be solved by inverse scattering transforms. We note that for NEE (1.1) associated with isospectral deformations of (1.10), the recursion operator ϕ is a strong symmetry, i.e. it satisfies (2.2), or equivalently

$$\phi_t=[K_l'\phi]\qquad\text{if}\qquad u_t=K_l.$$

But for NEE (2.5), ϕ satisfies (2.4), or (with $k=1$, $n=N+1$ and $\tau=\tau_{N+1}^l$ in (2.4))

$$\phi_t=[\tau'\phi]+\phi^{N+1}\qquad\text{if}\qquad u_t=\tau.$$

It would be preferable to replace the time variant parameter ξ ($\xi_t=\xi^{N+1}$) by an invariant one $\bar{\xi}=f(\xi,t)=\xi(1+Nt\xi^N)^{-1/N}(\bar{\xi}_t=\partial f/\partial t+(\partial f/\partial\xi)\xi_t=0)$ and then naturally consider $\bar{\phi}=f(\phi,t)$ as a strong symmetry for NEE (2.5). We proved this idea in § 3 and, what is more, we found that the expressions of constants of motion $\bar{I}_n$ in terms of scattering data in (4.7) are agreeable to our replacement, namely (4.7) can be obtained by using $f(\xi,t)$ to replace ξ, etc, in (4.6).

We also note that for $N=-1$, we have θ grad $\bar{I}_n=\bar{K}_n$. This means that symmetries $\bar{K}_n$ yield the constants of motion. But for $N\geqslant 0$, unlike most integrable Hamiltonian systems, symmetries $\bar{K}_n$ no longer yield constants of motion for NEE (2.5), although they still correspond to gradient functions through the impletic operator θ, while θ grad $\bar{I}_n$ are not symmetries.

Acknowledgments

One of the authors (YL) would like to thank R K Bullough for the invitation to visit UMIST and the British Council for hospitality during the visit. The work was partially supported by the National Science Fund of The People's Republic of China.

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